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3. The abstract of this paper reads as follows:

"The paper deals with the investigation of elastic waves in an infinite homogenous, isotropic and perfectly elastic medium produced by a spherical source on the assumption that the spheroidally distributed stress on the surface of the source is an arbitrary function of time. The case of an explosive source leading to an exciting function of the shock type is investigated and discussed in detail. For waves having orders n = 0 and n = 1 the dependence of the maximum amplitude on the distance is determined. The distance up to which the manner of propagation of the considered waves is affected by the mechanism of their generation is also determined."

4. This paper represents scientific capability in mathematics and geophysics. It deals with the condition of shock waves near the source, in this respect constituting an extension of previous US research. It appears to be concerned with applied geophysics (i.e. seismic exploration; artificial explosions). Theoretically, no

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A CONTRIBUTION TO THE THEORY OF ELASTIC WAVES PRODUCED BY SHOCK

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The paper deals with the investigation of elastic waves in an infinite homogenous, isotropic and perfectly elastic medium produced by a spherical source on the assumption that the spheroidally distributed stress on the surface of the source is an arbitrary function of time. The case of an explosive source leading to an exciting function of the shock type is investigated and discussed in detail. For waves having orders $n = 0$ and $n = 1$ the dependence of the maximum amplitude on the distance is determined. The distance up to which the manner of propagation of the considered waves is affected by the mechanism of their generation is also determined.

I. INTRODUCTION

This paper is the first of a series which are to be devoted to the influence of the mechanism of generation of elastic waves on the manner of their propagation. Elastic waves due to a spherical source are investigated on the assumption that the spheroidally distributed stress on the surface of the source is generally an arbitrary function of time. This function is called an *exciting function*. Special attention is paid to elastic waves generated by a shock exciting function $f(t) = \sigma t^{\nu} e^{-\alpha t}$ which may be of certain importance in applied (exploring) seismology.

The method of series introduced into the theory of elastic waves by Sezawa [1] was used for the solution of the basic equations, the problem being further reduced by means of the Laplace transformation to the solution of Bromwich-Wagner integrals. The paper starts from the theory of elastic waves due to a spherical source as developed by the Japanese seismic school for harmonic waves as well as for some simple exciting functions. A survey of papers on this subject published up to 1936 was given by Kawasumi [2]; the more important of the recent papers are those of Nishimura [3] and Sezawa and Kanai [4].

II. THE FORMULATION OF THE PROBLEM AND ITS SOLUTION

Let us assume an infinite, homogeneous, isotropic and perfectly elastic medium characterized by density ϱ and the Lamé parameters λ, μ. Let there be in this medium a spherical cavity of radius a with its centre at the origin and let this cavity be the source of elastic waves. Further let r, φ, ϑ be spherical coordinates and u_1, u_2, u_3 the corresponding components of the vector of elastic displacement. These components of displacement are to be determined at an arbitrary point of the medium adjacent to the source for any time t on the assumption that the spheroidally[1] distributed stress on the surface of the cavity is in general an arbitrary function of time. The solution of the problem thus formulated is too difficult and we shall therefore neglect the component of displacement u_2, and shall assume that u_1 and u_3 are independent of φ.

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[1]) i. e. according to Legendre polynomials

By this the problem is reduced to two dimensions in the plane ϑ = const. If in addition we neglect external forces[2]) we obtain for the propagation of dilatational and shear waves the fundamental equations

$$\frac{\varrho}{\lambda + 2\mu} \frac{\partial^2 \Theta}{\partial t^2} = \frac{\partial^2 \Theta}{\partial r^2} + \frac{2}{r} \frac{\partial \Theta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Theta}{\partial \varphi^2} + \frac{1}{r^2} \frac{\partial \Theta}{\partial \varphi} \cot \varphi \,, \tag{1a}$$

$$\frac{\varrho}{\mu} \frac{\partial^2 \omega}{\partial t^2} = \frac{\partial^2 \omega}{\partial r^2} + \frac{2}{r} \frac{\partial \omega}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \omega}{\partial \varphi^2} + \frac{1}{r^2} \frac{\partial \omega}{\partial \varphi} \cot \varphi - \frac{\omega}{r^2} (1 + \cot^2 \varphi) \tag{1b}$$

respectively, where

$$\Theta = \frac{\partial u_1}{\partial r} + \frac{2 u_1}{r} + \frac{1}{r} \frac{\partial u_2}{\partial \varphi} + \frac{u_2}{r} \cot \varphi \,, \tag{2a}$$

$$2\omega = \frac{\partial u_2}{\partial r} + \frac{u_2}{r} - \frac{1}{r} \frac{\partial u_1}{\partial \varphi} \,. \tag{2b}$$

The problem can now be formulated as follows:

The functions $u_1(r, \varphi, t)$ and $u_2(r, \varphi, t)$ are to be determined satisfying

A. the fundamental equations (1a, 1b),

B. the boundary conditions on the surface of the sphere

$$\begin{aligned} p_{rr} &= f(t)\, P_n(\cos \varphi) \,, \\ p_{r\varphi} &= \chi(t) \frac{\mathrm{d} P_n(\cos \varphi)}{\mathrm{d}\varphi} \end{aligned} \tag{3}$$

valid for $r = a$; p_{rr} being the radial and $p_{r\varphi}$ the tangential components of stress, $P_n(\cos \varphi)$ the Legendre polynomials of order n, and $f(t)$, $\chi(t)$ being arbitrary functions of time.

C. the initial conditions for $t = 0$

$$\begin{aligned} u_1 &= 0 \,, \quad \dot{u}_1 = 0 \,, \\ u_2 &= 0 \,, \quad \dot{u}_2 = 0 \,. \end{aligned} \tag{4}$$

In addition we shall require

D. that the functions u_1, u_2 for $r \to \infty$ should be finite or zero.

Let us substitute in Equ. (1a, b)

$$\Theta = \frac{1}{2\pi \mathrm{i}} \int_{c - \mathrm{i}\infty}^{c + \mathrm{i}\infty} \frac{e^{zt}}{z} \Phi \, \mathrm{d}z \,, \tag{5a}$$

$$2\omega = \frac{1}{2\pi \mathrm{i}} \int_{c - \mathrm{i}\infty}^{c + \mathrm{i}\infty} \frac{e^{zt}}{z} \Psi \, \mathrm{d}z \,, \tag{5b}$$

where the integral $\int_{c - \mathrm{i}\infty}^{c + \mathrm{i}\infty} \ldots \mathrm{d}z$ is the Bromwich-Wagner integral in the complex z-plane and Φ and Ψ are functions of the complex variable z.

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[2]) Possibly gravity could be considered but its effect on the bodily waves as shown by JEFFREYS [5] is negligible.

[illegible] conditions (4) we obtain

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{1}{r^2}\frac{\partial \Phi}{\partial \varphi}\cot\varphi - \frac{\varrho}{\lambda + 2\mu} z^2 \Phi = 0\,, \tag{6a}$$

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{2}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \varphi^2} + \frac{1}{r^2}\frac{\partial \Psi}{\partial \varphi}\cot\varphi - \frac{\Psi}{r^2}(1 + \cot^2\varphi) - \frac{\varrho}{\mu} z^2 \Psi = 0\,. \tag{6b}$$

The solution of these equations with regard to condition D is

$$\Phi = A_n r^{-1/2} H^{(1)}_{n+1/2}(\mathrm{i}z\beta_1 r)\, P_n(\cos\varphi)\,, \tag{7a}$$

$$\Psi = B_n r^{-1/2} H^{(1)}_{n+1/2}(\mathrm{i}z\beta_2 r)\, \frac{\mathrm{d}P_n(\cos\varphi)}{\mathrm{d}\varphi}\,, \tag{7b}$$

where

$$\beta_1 = (\varrho/\lambda + 2\mu)^{1/2}\,, \quad \beta_2 = (\varrho/\mu)^{1/2} \tag{7'}$$

and $H^{(1)}_{n+1/2}$ is the Hankel function of the first kind.

The dilatational components of displacement $u_j^{(1)}$ *) following from the solution of Equ. (7a) for Φ and satisfying the condition $\omega = 0$ fulfil the relations

$$u_1^{(1)} = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{zt}}{z}\, \frac{A_n}{z^2\beta_1^2}\, \frac{\mathrm{d}}{\mathrm{d}r} \frac{H^{(1)}_{n+1/2}(\mathrm{i}z\beta_1 r)}{r^{1/2}}\, P_n(\cos\varphi)\, \mathrm{d}z\,, \tag{8a}$$

$$u_2^{(1)} = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{zt}}{z}\, \frac{A_n}{z^2\beta_1^2}\, \frac{H^{(1)}_{n+1/2}(\mathrm{i}z\beta_1 r)}{r^{3/2}}\, \frac{\mathrm{d}P_n(\cos\varphi)}{\mathrm{d}\varphi}\, \mathrm{d}z\,. \tag{8b}$$

The shear components of displacement $u_j^{(2)}$ following from the solution of Equ. (7b) for Ψ and satisfying the condition $\Theta = 0$ are given by the expressions

$$u_1^{(2)} = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{zt}}{z}\, \frac{n(n+1)\, B_n}{z^2\beta_2^2}\, \frac{H^{(1)}_{n+1/2}(\mathrm{i}z\beta_2 r)}{r^{3/2}}\, P_n(\cos\varphi)\, \mathrm{d}z\,, \tag{8c}$$

$$u_2^{(2)} = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{zt}}{z}\, \frac{B_n}{z^2\beta_2^2}\, \frac{1}{r}\, \frac{\mathrm{d}}{\mathrm{d}r} \{r^{1/2} H^{(1)}_{n+1/2}(\mathrm{i}z\beta_2 r)\}\, \frac{\mathrm{d}P_n(\cos\varphi)}{\mathrm{d}\varphi}\, \mathrm{d}z\,. \tag{8d}$$

It is evident that

$$u_j = u_j^{(1)} + u_j^{(2)}\,.$$

A_n and B_n are determined from the boundary conditions (3). For the normal component of stress at $r = a$ the following holds

$$p_{rr} = \left\{\lambda\Theta + 2\mu \frac{\partial u_1}{\partial r}\right\}_{r=a} = f(t)\, P_n(\cos\varphi)\,, \tag{9a}$$

while for the tangential component we get

$$p_{r\varphi} = \left\{\mu\left(\frac{\partial u_2}{\partial r} - \frac{u_2}{r} + \frac{1}{r}\frac{\partial u_1}{\partial \varphi}\right)\right\}_{r=a} = \chi(t)\, \frac{\mathrm{d}P_n(\cos\varphi)}{\mathrm{d}\varphi}\,. \tag{9b}$$

*) The indices in [illegible] expressions $u_j^{(i)}$ have the values $i, j = 1, 2$ in this paper.

Using the inversion theorem of the Laplace transformation [7] we can express the function $f(t)$ in the form

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{zt}}{z}\, g(z)\, \mathrm{d}z\,, \tag{10a}$$

where $g(z)$ is the image of the original function $f(t)$ defined by the expression

$$g(z) = z \int\limits_0^\infty \mathrm{e}^{-zt} f(t)\, \mathrm{d}t\,. \tag{11a}$$

Similarly

$$\chi(t) = \frac{1}{2\pi \mathrm{i}} \int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{zt}}{z}\, \gamma(z)\, \mathrm{d}z\,, \tag{10b}$$

where

$$\gamma(z) = z \int\limits_0^\infty \mathrm{e}^{-zt} \chi(t)\, \mathrm{d}t\,. \tag{11b}$$

Using Equ. (10a), Equ. (9a) can now be written

$$p_{rr} = \frac{P_n(\cos\varphi)}{2\pi \mathrm{i}} \int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{zt}}{z}\, g(z)\, \mathrm{d}z\,. \tag{12a}$$

Similarly with the help of (10b) Equ. (9a) can now be written

$$p_{r\varphi} = \frac{\mathrm{d}P_n(\cos\varphi)}{\mathrm{d}\varphi}\, \frac{1}{2\pi \mathrm{i}} \int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{zt}}{z}\, \gamma(z)\, \mathrm{d}z\,. \tag{12b}$$

By substituting (5a) and (8a, b, c, d) into (12a, b) we get from (12a)

$$\frac{P_n(\cos\varphi)}{2\pi \mathrm{i}} \int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{zt}}{z} \left\{ \frac{2\mu M_n}{a^{3/2} z^2 \beta_1^2}\, A_n + \frac{2\mu(n+1)\, n K_n}{a^{3/2} z^2 \beta_2^2}\, B_n \right\} \mathrm{d}z = \frac{P_n(\cos\varphi)}{2\pi \mathrm{i}} \int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{zt}}{z}\, g(z)\, \mathrm{d}z \tag{13a}$$

and from (12b)

$$\frac{\mathrm{d}P_n(\cos\varphi)}{\mathrm{d}\varphi}\, \frac{1}{2\pi \mathrm{i}} \int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{zt}}{z} \left\{ \frac{2\mu L_n}{a^{3/2} z^2 \beta_1^2}\, A_n + \frac{\mu N_n}{a^{3/2} z^2 \beta_2^2}\, B_n \right\} \mathrm{d}z = \frac{\mathrm{d}P_n(\cos\varphi)}{\mathrm{d}\varphi}\, \frac{1}{2\pi \mathrm{i}} \int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{zt}}{z}\, \gamma(z)\, \mathrm{d}z\,. \tag{13b}$$

From this

$$A_n = \frac{g(z) + 2n(n+1)\, \dfrac{K_n}{N_n}\, \gamma(z)}{\Omega_n}\,, \tag{14a}$$

$$B_n = 2 \left(\frac{\beta_2}{\beta_1} \right)^2 \frac{L_n g(z) - M_n \gamma(z)}{N_n \Omega_n}\,, \tag{14b}$$

where we introduced the abbreviations

$$\Omega_n = \frac{2\mu}{a^{3/2} \beta_1^2 z^2 N_n} \{ M_n N_n + 2n(n+1)\, K_n L_n \}\,, \tag{15a}$$

$$K_n = (n-1)\, H^{(1)}_{n+1/2}(\mathrm{i}z\beta_2 a) - \mathrm{i}z\beta_2 a\, H^{(1)}_{n+3/2}(\mathrm{i}z\beta_2 a)\,, \tag{15b}$$

$$L_n = (n - 1) H^{(1)}_{n+1/2}(iz\beta_2 a) - iz\beta_2 a H^{(1)}_{n+3/2}(iz\beta_2 a)$$

$$M_n = \left\{\frac{\lambda}{2\mu}(z\beta_1 a)^2 + n(n+1)\right\} H^{(1)}_{n+1/2}(iz\beta_1 a) - \tag{15d}$$
$$- (2n+1)\, iz\beta_1 a H^{(1)}_{n+3/2}(iz\beta_1 a) - (z\beta_1 a)^2 H^{(1)}_{n+5/2}(iz\beta_1 a)\,,$$

$$N_n = - 2(n^2 - 1) H^{(1)}_{n+1/2}(iz\beta_2 a) + (2n+1)\, iz\beta_2 a H^{(1)}_{n+3/2}(iz\beta_2 a) + \tag{15e}$$
$$+ (z\beta_2 a)^2 H^{(1)}_{n+5/2}(iz\beta_2 a)\,.$$

Substituting expressions (14a, b) into (5a, b) and (8a — d) we finally obtain

$$\Theta = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{zt}}{z} \frac{g(z) + 2n(n+1)\dfrac{K_n}{N_n}\gamma(z)}{\Omega_n} \frac{H^{(1)}_{n+1/2}(iz\beta_1 r)}{r^{1/2}} P_n(\cos\varphi)\, dz\,, \tag{16a}$$

$$2\omega = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{zt}}{z} \frac{2(L_n g(z) - M_n \gamma(z))}{N_n \Omega_n} \left(\frac{\beta_2}{\beta_1}\right)^2 \frac{H^{(1)}_{n+1/2}(iz\beta_2 r)}{r^{1/2}} \frac{dP_n(\cos\varphi)}{d\varphi}\, dz \tag{16b}$$

and for the components of displacement

$$u_1^{(1)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{zt}}{z} \frac{g(z) + 2n(n+1)\dfrac{K_n}{N_n}\gamma(z)}{z^2\beta_1^2\Omega_n} \frac{d}{dr} \frac{H^{(1)}_{n+1/2}(iz\beta_1 r)}{r^{1/2}} P_n(\cos\varphi)\, dz\,, \tag{17a}$$

$$u_2^{(1)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{zt}}{z} \frac{g(z) + 2n(n+1)\dfrac{K_n}{N_n}\gamma(z)}{z^2\beta_1^2\Omega_n} \frac{H^{(1)}_{n+1/2}(iz\beta_1 r)}{r^{3/2}} \frac{dP_n(\cos\varphi)}{d\varphi}\, dz\,, \tag{17b}$$

$$u_1^{(2)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{zt}}{z} \frac{2n(n+1)\,\{L_n g(z) - M_n \gamma(z)\}}{z^2\beta_1^2 N_n \Omega_n} \frac{H^{(1)}_{n+1/2}(iz\beta_2 r)}{r^{3/2}} P_n(\cos\varphi)\, dz\,, \tag{17c}$$

$$u_2^{(2)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{zt}}{z} \frac{2(L_n g(z) - M_n \gamma(z))}{z^2\beta_1^2 N_n \Omega_n} \frac{1}{r} \frac{d}{dr}\{r^{1/2} H^{(1)}_{n+1/2}(iz\beta_2 r)\} \frac{dP_n(\cos\varphi)}{d\varphi}\, dz\,. \tag{17d}$$

The images $g(z)$ and $\gamma(z)$ have been calculated for a number of functions $f(t)$ and $\chi(t)$ [6, 7], we can therefore assume them to be known. It remains therefore to complete the solution of the problem to carry out the integration of the **Bromwich-Wagner integrals** in (16a, b) and (17a — d).

III. SOLUTION FOR SHOCK-PRODUCED WAVES

In the present paper we shall consider source of explosive character, i. e. that the boundary conditions (3) on the surface of the spherical source shall be specialized by assuming $\chi(t) = 0$. This reduces our problem to purely radial stress or pressure which is the case in practice with explosions. The boundary conditions then take the form

$$p_{rr} = - f(t)\, P_n(\cos\varphi)\,, \quad p_{r\varphi} = 0\ ^{4)} \tag{3'}$$

[4]) The minus sign expresses the fact that the pressure points in the opposite direction of the internal normal of the sphere.

and the components of displacement Equ. (17a — d) reduce to

$$u_1^{(1)} = \frac{-1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{e^{zt}}{z^3\beta_1^2}\,\frac{g(z)}{\Omega_n}\,\frac{d}{dr}\,\frac{H^{(1)}_{n+1/2}(iz\beta_1 r)}{r^{1/2}}\,P_n(\cos\varphi)\,dz\,, \tag{17a'}$$

$$u_2^{(1)} = \frac{-1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{e^{zt}}{z^3\beta_1^2}\,\frac{g(z)}{\Omega_n}\,\frac{H^{(1)}_{n+1/2}(iz\beta_1 r)}{r^{3/2}}\,\frac{dP_n(\cos\varphi)}{d\varphi}\,dz\,, \tag{17b'}$$

$$u_1^{(2)} = \frac{-1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{e^{zt}}{z^3\beta_1^2}\,\frac{2n(n+1)\,L_n\,g(z)}{N_n\Omega_n}\,\frac{H^{(1)}_{n+1/2}(iz\beta_2 r)}{r^{3/2}}\,P_n(\cos\varphi)\,dz\,, \tag{17c'}$$

$$u_2^{(2)} = \frac{-1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{e^{zt}}{z^3\beta_1^2}\,\frac{2L_n\,g(z)}{N_n\Omega_n}\,\frac{1}{r}\,\frac{d}{dr}\{r^{1/2}H^{(1)}_{n+1/2}(iz\beta_2 r)\}\,\frac{dP_n(\cos\varphi)}{d\varphi}\,dz\,. \tag{17d'}$$

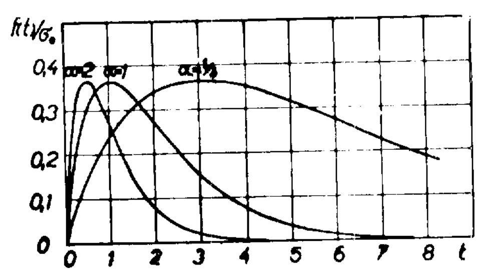

Fig. 1a. Time dependence of the function $\sigma_0\alpha^\nu t^\nu e^{-\alpha t}$ for different values of α, $\nu = 1$.

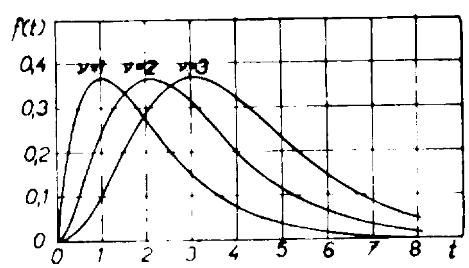

Fig. 1b. Time dependence of the function $\sigma_\nu t^\nu e^{-\alpha t}$ for different values of ν; $\alpha = 1$, $\sigma_1 = 1$, $\sigma_2 = 0{\cdot}680$, $\sigma_3 = 0{\cdot}274$.

There still remains to determine the character of the exciting function $f(t)$. To approximate actual conditions during explosions as well as possible let us choose as the exciting function a fairly general shock function [8]

$$f(t) = 0\,, \qquad t < 0\,,$$
$$f(t) = \sigma t^\nu e^{-\alpha t}\,, \quad t > 0\,, \tag{18a}$$

where σ, $\alpha > 0$ are real parameters, ν a non-negative integer. $f(t)$ versus t for various values of α and ν is plotted in Fig. 1a, b.

The image of the function $f(t)$ is the function [6]

$$g(z) = 0\,,$$
$$g(z) = \frac{\nu!\sigma z}{(z+\alpha)^{\nu+1}}\,, \tag{18b}$$

for which in the following waves having orders $n = 0$ and $n = 1$[5]) will be investigated.

[5]) Waves of order $n = 2$ will be the subject of the second part of the paper to be published in the future.

Expressions (15a — e) reduce to the form

$$\Omega_0 = \frac{2\mu}{a^4 z^2 \beta_1^2} M_0 ,$$

$$M_0 = -\mathrm{i}\left(\frac{2}{\pi \mathrm{i} z \beta_1}\right)^{1/2} \mathrm{e}^{-az\beta_1} a^{-1/2} \Omega_{00} ,$$

where
$$\Omega_{00} = (1 + \lambda/2\mu)(z\beta_1 a)^2 + 2(z\beta_1 a) + 2.$$

All the components of displacement excepting $u_1^{(1)}$ are equal to zero. We thus have the case of spherical waves with a dilatational (radial) component of displacement. For $u_1^{(1)}$ we can write u_1 and the integral (17a') becomes after substituting the corresponding quantities and rearrangement

$$u_1 = \frac{1}{2\pi \mathrm{i}} \frac{a^3}{2\mu r^2} \int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(1 + z\beta_1 r)\exp\{z[t - \beta_1(r - a)]\}\, g(z)}{z\Omega_{00}} \mathrm{d}z . \qquad (19)$$

Introducing the image (18b) of the chosen function into (19) we obtain for $[t - \beta_1(r - a)] > 0$

$$u_1 = \frac{1}{2\pi \mathrm{i}} \frac{\nu!\sigma a^3}{2\mu r^2} \int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(1 + z\beta_1 r)\exp\{z[t - \beta_1(r - a)]\}}{(z + \varkappa)^{\nu+1}\,\Omega_{00}} \mathrm{d}z . \qquad (20)$$

The integral (20) was evaluated using the path customary in the evaluation of Bromwich-Wagner integrals [7]. The integrand has simple poles at the points $z_{1,2} = (a\beta_1)^{-1}(\omega \pm \mathrm{i}\omega')$ where $\omega = -(1 + \lambda/2\mu)^{-1}$, $\omega' = (1 + \lambda/2\mu)^{-1} \cdot (1 + \lambda/\mu)^{1/2}$ which are the roots of the equation

$$\Omega_{00}(z) = 0 \qquad (21)$$

and a $(\nu + 1)$-multiple pole at the point $z_3 = -\varkappa$.

Introducing the abbreviations

$$\xi = a\beta_1\varkappa, \quad R = \frac{r}{a}, \quad T_i = t - \beta_i(r - a), \tau_i = \frac{T_i}{a\beta_1}, \qquad (22)$$

we obtain for the displacement (20) the expression

$$u_1 = \frac{a\sigma\nu!(a\beta_1)^\nu}{2\mu(1 + \lambda/2\mu) R^2}\left\{\mathrm{e}^{-\xi\tau_1} \sum_{k=0}^{\nu} \sum_{l=0}^{k} \left[\frac{(-1)^k (a\beta_1)^k}{(\nu - k)!} \frac{(1 - \xi R)\tau_1^{\nu-k} + (\nu - k) R\tau_1^{\nu-k-1}}{[(\omega + \xi)^2 + \omega'^2]^{k+1}} \cdot \right.\right.$$

$$\left. \cdot (-\varkappa - z_1)^l (-\varkappa - z_2)^{k-l}\right] + \frac{1}{\omega'} \frac{[(1 + \omega R)^2 + (\omega' R)^2]^{1/2}}{[(\omega + \xi)^2 + \omega'^2]^{1/2(\nu+1)}} \mathrm{e}^{\omega\tau_1} . \qquad (23)$$

$$\left. . \sin\left(\omega'\tau_1 + \operatorname{arctg}\frac{\omega' R}{1 + \omega R} - (\nu + 1)\operatorname{arctg}\frac{\omega'}{\omega + \xi}\right)\right\}, \qquad T_1 > 0,$$

$$u_1 = 0, \qquad T_1 < 0.$$

This expression represents a travelling wave produced by a spherical source for the given exciting function. The first term in the curled brackets, which is due to the pole at the point $z = -\varkappa$ represents the forced portion of the wave,*) while the second term which is due to poles satisfying Equ. (21) represents the free wave.

*) See Chapter III para 3.

Choosing especially $\nu = 1$ reduces equation (23) to

$$u_1 = \frac{a^2\beta_1\sigma}{2\mu(1+\lambda/2\mu)\,R^2}\left\{\left[\frac{(1-\xi R)\,\tau_1 + R}{[(\omega+\xi)^2+\omega'^2]} + \frac{2(1-\xi R)(\omega+\xi)}{[(\omega+\xi)^2+\omega'^2]^2}\right]e^{-\xi\tau_1} + \right.$$
$$\left. + \frac{1}{\omega'}\frac{[(1+\omega R)^2+(\omega' R)^2]^{1/2}}{[(\omega+\xi)^2+\omega'^2]}e^{\omega\tau_1}\sin\left(\omega'\tau_1 + \operatorname{arctg}\frac{\omega' R}{1+\omega R} - 2\operatorname{arctg}\frac{\omega'}{\omega+\xi}\right)\right\},$$
$$T_1 > 0\,, \qquad (24)$$
$$u_1 = 0\,, \qquad T_1 < 0\,.$$

At a considerable distance from the source, i. e. for $r \gg a$ no great error is committed when calculating the displacement u_1 if only terms having the lowest power of $1/r$ are considered, higher power terms being neglected. Then

$$u_1 = \frac{a^2\beta_1\sigma}{2\mu(1+\lambda/2\mu)\,R}\left\{\left[\frac{1-\xi\tau_1}{[(\omega+\xi)^2+\omega'^2]} - \frac{2\xi(\omega+\xi)}{[(\omega+\xi)^2+\omega'^2]^2}\right]e^{-\xi\tau_1} + \right.$$
$$\left. + \frac{1}{\omega'}\frac{(\omega^2+\omega'^2)^{1/2}}{[(\omega+\xi)^2+\omega'^2]}e^{\omega\tau_1}\sin\left(\omega'\tau_1 + \operatorname{arctg}\frac{\omega'}{\omega} - 2\operatorname{arctg}\frac{\omega'}{\omega+\xi}\right)\right\}, \qquad (25)$$
$$T_1 > 0\,,$$
$$u_1 = 0\,, \qquad T_1 < 0\,.$$

The time dependence of the free and forced waves and the curve of the resulting displacement for the parameters $\nu = 1$, $\xi = 1$ for $r = a$ and $r \gg a$ is shown in Fig. 2 and 3 respectively on the assumption $\lambda = \mu$ which for rocks in the earth's crust is reasonably satisfied [9]. The resulting displacement for $r = a$ and for $r \gg a$ starts from its rest position since for $\tau_1 = 0$ the free and forced waves cancel. Even though the exciting function

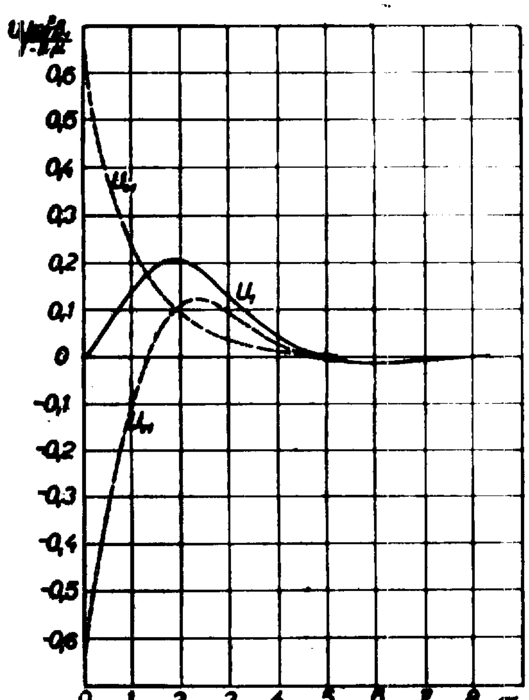

Fig. 2. Waves of order $n = 0$. The curve of the displacement u_1 for $r = a$; $\nu = 1$, $\xi = 1$. $u_{\varphi 1}$ forced, $u_{\nu 1}$ free part.

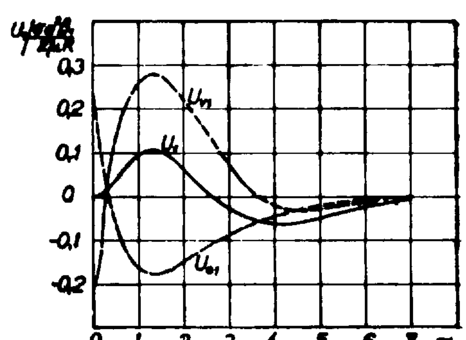

Fig. 3. Waves of order $n = 0$. The curve of the displacement u_1 for $r \gg a$; $\nu = 1$, $\xi = 1$, $u_{\varphi 1}$ forced, $u_{\nu 1}$ free part.

is *aperiodic* the resulting wave is a damped *periodic* wave. Because of the factor $e^{-\xi\tau_1}$ the damping of the forced wave depends on the parameter ξ, while the damping and frequency of the characteristic wave depends on the roots of equation (21) about the physical meaning of which we shall deal in more detail in para. 3 of this chapter. Figs. 2 and 3 represent of course the time dependence of the displacement only for two limiting cases ($r = a$ and $r \gg a$).

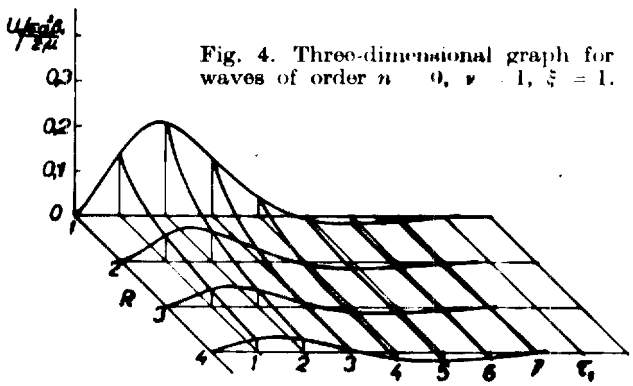

Fig. 4. Three-dimensional graph for waves of order $n = 0$, $\nu = 1$, $\xi = 1$.

The dependence of the displacement u_1 on both the time variable τ_1 and the distance r from the source (i. e. the reduced distance R) is fully expressed by representing the wave motion in a three-dimensional diagram. The three-dimensional (Fig. 4) for $\nu = 1, \xi = 1$ shows that the amplitude of the periodic waves decrease with increasing distance according to a certain law[7]) without the character of the wave changing. The damping of the resulting wave also decreases with increasing distance. As against the originally strongly damped waves in the vicinity of the source the wave tends to become undamped at greater distances from the source.

2. Waves of Order $n = 1$.

Let us assume $\lambda = \mu$ which implies $\beta_2 = \sqrt{3}\beta_1$. Expression (15b — c) will then be of the form

$$K_1 = \left(\frac{2}{\pi i z\beta_2}\right)^{1/2} \frac{e^{-az\beta_2}}{z\beta_2 a^{1/2}} \{(z\beta_2 a)^2 + 3(z\beta_2 a) + 3\},$$

$$L_1 = \left(\frac{2}{\pi i z\beta_2}\right)^{1/2} \frac{e^{-az\beta_1}}{3^{1/2} z\beta_2 a^{1/2}} \{(z\beta_2 a)^2 + 3\sqrt{3}(z\beta_2 a) + 9\},$$

$$M_1 = -\left(\frac{2}{\pi i z\beta_2}\right)^{1/2} \frac{3^{1/2}\, e^{-az\beta_1}}{6z\beta_2 a^{1/2}} \{3(z\beta_2 a)^3 + 7\sqrt{3}(z\beta_2 a)^2 + 36(z\beta_2 a) + 36\sqrt{3}\},$$

$$N_1 = \left(\frac{2}{\pi i z\beta_2}\right)^{1/2} \frac{e^{-az\beta_2}}{z\beta_2 a^{1/2}} \{(z\beta_2 a)^3 + 3(z\beta_2 a)^2 + 6(z\beta_2 a) + 6\}$$

and expression (15a)

$$\Omega_1 = -\frac{3^{1/2}\mu(2/\pi i z\beta_2)^{1/2}\, e^{-az\beta_1}}{z\beta_2 a^2} \frac{\Omega_{01}}{\sigma_1},$$

where $$\Omega_{01} = (z\beta_2 a)^4 + (3 + 7/\sqrt{3})(z\beta_2 a)^3 + (18 + 13/\sqrt{3})(z\beta_2 a)^2 + (18 + 18\sqrt{3})(z\beta_2 a) + 18\sqrt{3}$$

and $$\sigma_1 = (z\beta_2 a)^3 + 3(z\beta_2 a)^2 + 6(z\beta_2 a) + 6.$$

[7]) According to the law investigated in chapter IV para 1.

Substituting the proper expressions into (17a — d') we obtain for the components of displacement ($T_i > 0$)

$$u_j^{(i)} = \frac{C_j^{(i)}}{2\pi \mathrm{i}} \frac{\nu!\sigma a^2}{\mu\beta_2^2 r^3} \int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{G_j^{(i)}(z)\exp\{z[t-\beta_i(r-a)]\}}{z^2(z+\alpha)^{\nu+1}\Omega_{01}(z)}\,\mathrm{d}z\,, \tag{26}$$

where

$$C_1^{(1)} = P_1(\cos\varphi)/\sqrt{3}\,, \qquad C_2^{(1)} = -\frac{\mathrm{d}P_1(\cos\varphi)}{\mathrm{d}\varphi}\,,$$

$$C_1^{(2)} = -4P_1(\cos\varphi)/\sqrt{3}\,, \qquad C_2^{(2)} = \frac{2}{\sqrt{3}}\frac{\mathrm{d}P_1(\cos\varphi)}{\mathrm{d}\varphi}\,,$$

$$G_1^{(1)}(z) = \{(z\beta_2 a)^3 + 3(z\beta_2 a)^2 + 6(z\beta_2 a) + 6\}\,.\,\{(z\beta_2 r)^2 + 2\sqrt{3}(z\beta_2 r) + 6\}\,.$$

$$G_1^{(2)}(z) = \{(z\beta_2 a)^2 + 3\sqrt{3}(z\beta_2 a) + 9\}\,.\,\{(z\beta_2 r) + 1\}\,,$$

$$G_2^{(1)}(z) = \{(z\beta_2 a)^3 + 3(z\beta_2 a)^2 + 6(z\beta_2 a) + 6\}\,.\,\{(z\beta_2 r) + \sqrt{3}\}\,,$$

$$G_2^{(2)}(z) = \{(z\beta_2 a)^2 + 3\sqrt{3}(z\beta_2 a) + 9\}\,.\,\{(z\beta_2 r)^2 + (z\beta_2 r) + 1\}\,.$$

Again the path of integration used is that customary when calculating Bromwich-Wagner integrals. The integrands have simple poles at points satisfying equation

$$\Omega_{01}(z) = 0\,, \tag{27}$$

a $(\nu+1)$-multiple pole at the point $z = -\alpha$ and a double pole at $z = 0$. The roots of Equ. (27) $z_{1,2} = (a\beta_2)^{-1}(\omega_1 \pm \mathrm{i}\omega_1')$, $z_3 = (a\beta_2)^{-1}\omega_2$, $z_4 = (a\beta_2)^{-1}\omega_3$ were determined by means of the values $\omega_1 = -1.58016$, $\omega_1' = 2.77111$, $\omega_2 = -2.77843$, $\omega_3 = -1.10272$ calculated to five decimal places by means of the Gräffe-Lobachevsky method [10]. The contributions from these poles correspond to the free wave, the contributions from other poles yield the forced wave. The portion of the displacement $u_j^{(i)}$ corresponding to the forced wave is denoted by $u_{oj}^{(i)} + u_{oj}^{(i)\prime}$ while the portion corresponding to the free wave is denoted by $u_{vj}^{(i)}$ so that

$$u_j^{(i)} = u_{oj}^{(i)} + u_{oj}^{(i)\prime} + u_{vj}^{(i)}\,. \tag{28}$$

Introducing in addition to the earlier abbreviations (22)

$$\eta = a\beta_2\alpha\,, \qquad \tau_i' = \frac{T_i}{a\beta_2}\,, \tag{29}$$

we get for the components of the forced wave after the integration of (26) the expressions

$$u_{oj}^{(i)} = \frac{\nu!\sigma(a\beta_2)^{\nu+1}D_j^{(i)}}{\mu\beta_2\eta^{\nu+1}R^3}\left\{\frac{\beta_2}{\beta_i}\left(\tau_i' - \frac{\nu+1}{\eta}\right) + R - 1\right\}, \qquad T_i > 0\,, \tag{30}$$

$$u_{oj}^{(i)} = 0\,, \qquad T_i < 0\,,$$

where

$$D_1^{(1)} = 2C_1^{(1)}\,3, \quad D_2^{(1)} = C_2^{(1)}\,3\sqrt{3}\,, \quad D_j^{(2)} = C_j^{(2)}/2\sqrt{3}$$

and further

$$u_{oj}^{(i)\prime} = \frac{\nu!\sigma C_j^{(i)}\,\mathrm{e}^{-\eta\tau_i'}}{\mu\beta_i(a\beta_2)^{\nu}R^{3}}\,B_j^{(i)}\,, \qquad T_i > 0\,, \tag{31}$$

$$u_{oj}^{(i)\prime} = 0\,, \qquad T_i < 0$$

where

$$B_j^{(i)} = \sum_{k_1=0}^{\nu}\left\{\sum_{k_2=0}^{k_1}\left[\frac{(-1)^{k_1}}{(\nu-k_1)!}\sum_{k_3=0}^{k_1-k_2}(-\varkappa-z_1)^{-(k_1-k_2-k_3+1)}(-\varkappa-z_2)^{-(k_3+1)}\,.\right.\right.$$

$$.\sum_{k_4=0}^{k_2}(-\varkappa-z_3)^{-(k_2-k_4+1)}(-\varkappa-z_4)^{-(k_4+1)}\Bigg]\sum_{k_5=0}^{\nu-k_1}(-1)^{k_5}(k_5+1)!\,(-\varkappa)^{-(k_5+2)}\,.$$

$$.\left.\sum_{k_6=0}^{\nu-k_1-k_5}\binom{\nu-k_1-k_5}{k_6}T_i^{\nu-k_1-k_5-k_6}\,G_j^{(i)\{k_6\}}(-\varkappa)\right\}.$$

The expression $G_j^{(i)\{k_6\}}(-\varkappa)$ is the k_6-th derivative of the expression $G_j^{(i)}(z)$ at the point $z=-\varkappa$. The waves $u_{oj}^{(i)}$ are similar to those which SEZAWA and KANAI [4] denoted by *forced waves of the first kind* and $u_{oj}^{(i)\prime}$ to those denoted by *forced waves of the second kind.*

By the integration of (26) we further obtain for the components of the free wave

$$u_{vj}^{(i)} = \frac{\nu!\sigma(\alpha\beta_2)^{\nu+1}C_j^{(i)}}{\mu\beta_2R^3}\left\{\sum_{k=2,3}\frac{G_{kj}^{(i)}e^{\omega_k\tau_i'}}{(\omega_k+\eta)^{\nu+1}\Gamma_k}+\frac{[(a_j^{(i)2}+b_j^{(i)2})(c_j^{(i)2}+d_j^{(i)2})]^{1/2}e^{\omega_1\tau_i'}}{\omega_1'[(\omega_1+\eta)^2+\omega_1'^2]^{1/2(\nu+1)}\Gamma_1^{1/2}}\,.\right.$$

$$\left.. \sin\left(\omega_1'\tau_i'-(\nu+1)\operatorname{arctg}\frac{\omega_1'}{\eta+\omega_1}+\gamma_j^{(i)}-\delta\right)\right\},\qquad T_i>0\,, \tag{32}$$

$$u_{vj}^{(i)}=0\,,\qquad T_i<0\,,$$

where

$$a_j^{(1)}=\omega_1^3+3\omega_1^2+6\omega_1+6-3\omega_1'^2(\omega_1+1)\,,$$
$$b_j^{(1)}=-\omega_1'^3+3\omega_1'(\omega_1^2+2\omega_1+2)\,,$$
$$c_1^{(1)}=(\omega_1^2-\omega_1'^2)R^2+2\sqrt{3}\omega_1R+6\,,\qquad c_2^{(1)}=\omega_1R+\sqrt{3}\,,$$
$$d_1^{(1)}=2\omega_1\omega_1'R^2+2\sqrt{3}\omega_1'R\,,\qquad d_2^{(1)}=\omega_1'R\,,$$

$$a_j^{(2)}=\omega_1^2+3\sqrt{3}\omega_1+9-\omega_1'^2\,,$$
$$b_j^{(2)}=\omega_1'(2\omega_1+3\sqrt{3})\,,$$
$$c_1^{(2)}=\omega_1R+1\,,\qquad c_2^{(2)}=(\omega_1^2-\omega_1'^2)R^2+\omega_1R+1\,,$$
$$d_1^{(2)}=\omega_1'R\,,\qquad d_2^{(2)}=2\omega_1\omega_1'R^2+\omega_1'R\,,$$
$$\gamma_j^{(i)}=\operatorname{arctg}\frac{b_j^{(i)}}{a_j^{(i)}}+\operatorname{arctg}\frac{d_j^{(i)}}{c_j^{(i)}}\,.$$

$$G_{k1}^{(1)}=(\omega_k^3+3\omega_k^2+6\omega_k+6)(\omega_k^2R^2+2\sqrt{3}\omega_kR+6)\,,\qquad k=2,3$$
$$G_{k2}^{(1)}=(\omega_k^3+3\omega_k^2+6\omega_k+6)(\omega_kR+\sqrt{3})\,,$$
$$G_{k1}^{(2)}=(\omega_k^2+3\sqrt{3}\omega_k+9)(\omega_kR+1)\,,$$
$$G_{k2}^{(2)}=(\omega_k^2+3\sqrt{3}\omega_k+9)(\omega_k^2R^2+\omega_kR+1)\,,$$
$$\Gamma_k=(-1)^k\omega_k^2(\omega_2-\omega_3)[(\omega_k-\omega_1)^2+\omega_1'^2]\,,$$
$$\Gamma_1=\prod_{h=1}^{3}(S_h^2+S_h'^2)\,,$$
$$\delta=\sum_{h=1}^{3}\operatorname{arctg}\frac{S_h'}{S_h}\,.$$

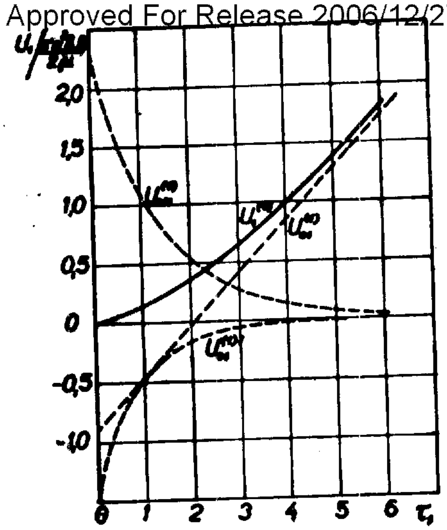

Fig. 5.

Fig. 5. Waves of order $n = 1$. Curve of the component $u_1^{(1)}$ for $r = a$; $\nu = 1$, $\xi = 1$.

Fig. 6. Waves of order $n = 1$. Curve of the component $u_1^{(2)}$ for $r = a$; $\nu = 1$, $\xi = 1$.

Fig. 7. Waves of order $n = 1$. Curve of the component $u_2^{(1)}$ for $r = a$; $\nu = 1$, $\xi = 1$.

Fig. 8. Waves of order $n = 1$. Curve of the component $u_2^{(2)}$ for $r = a$; $\nu = 1$, $\xi = 1$.

Fig. 9. Waves of order $n = 1$. The component of the resultant displacement u_1 for $r = a$; $\nu = 1$, $\xi = 1$.

Fig. 10. Waves of order $n = 1$. The component of the resultant displacement u_2 for $r = a$; $\nu = 1$, $\xi = 1$.

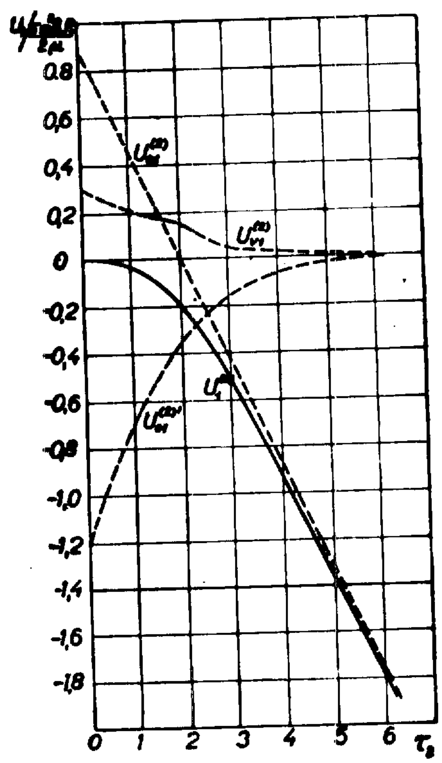

Fig. 6.

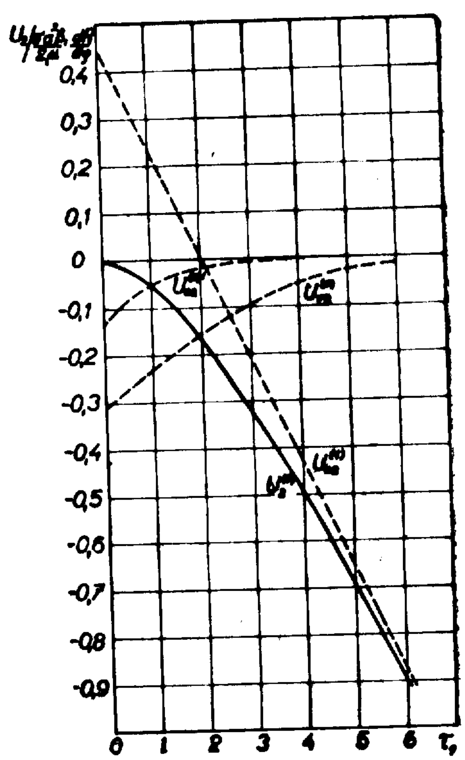

Fig. 7.

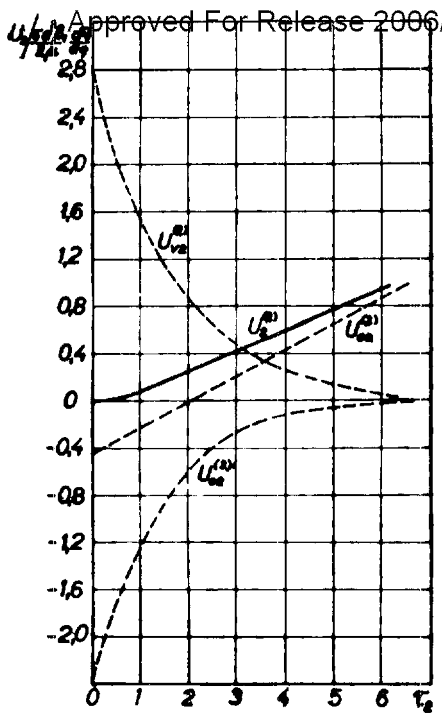

Fig. 8.

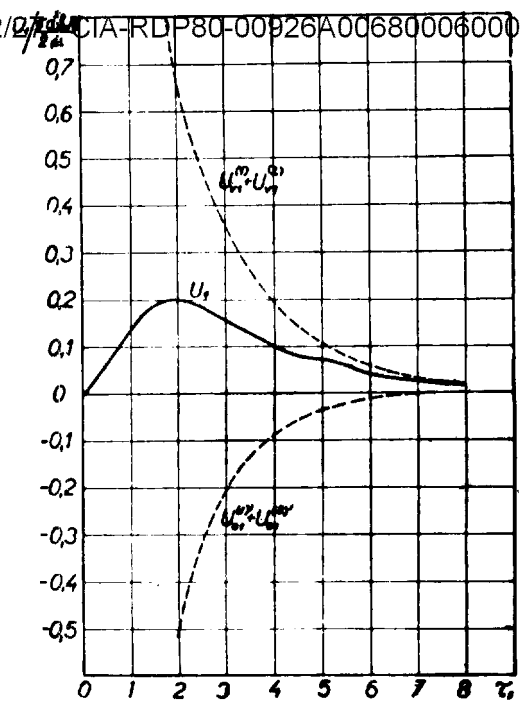

Fig. 9.

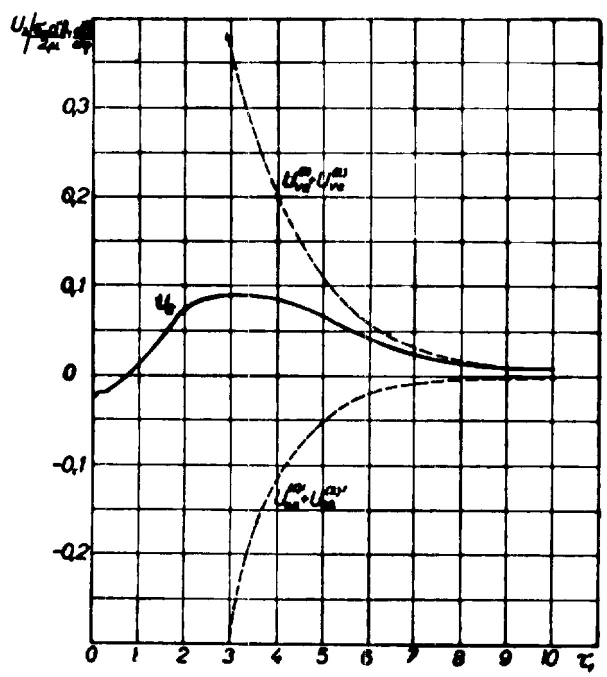

Fig. 10.

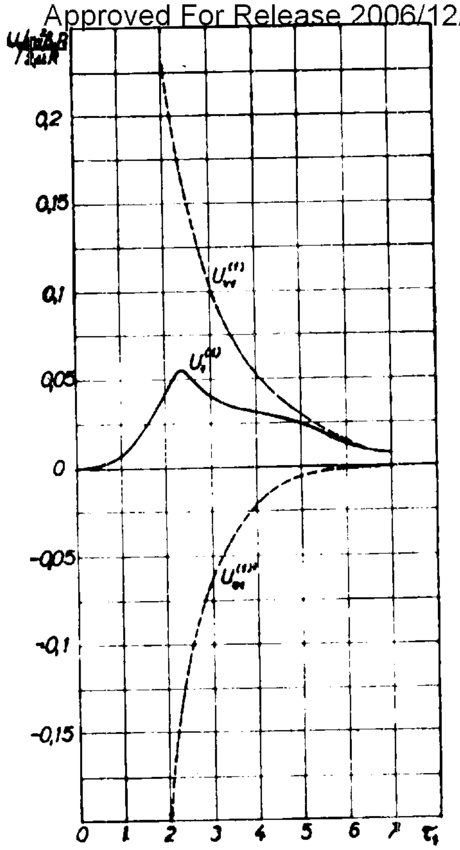

Fig. 11.

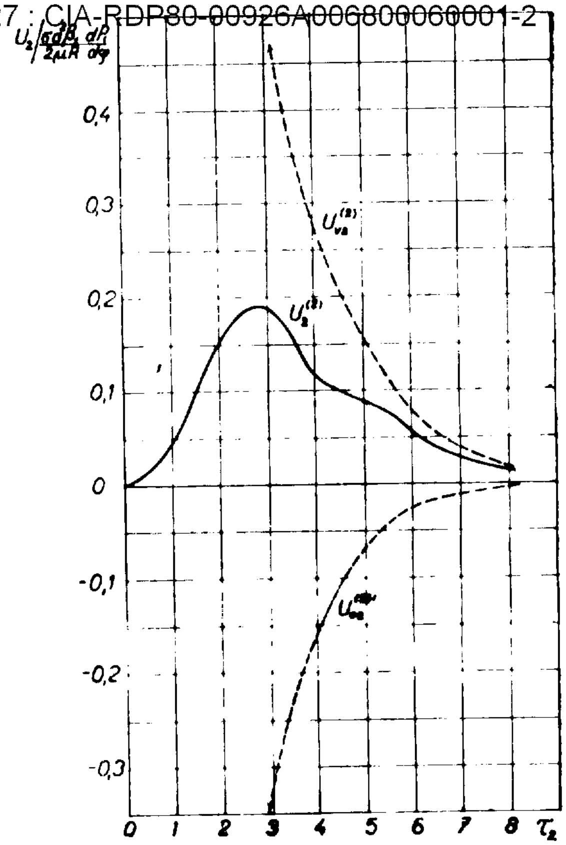

Fig. 12.

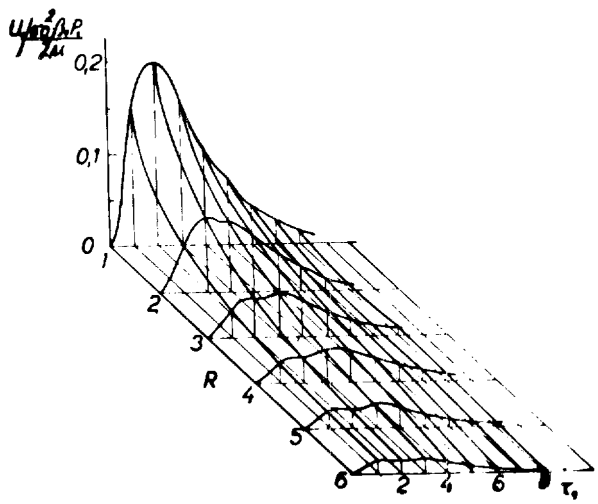

Fig. 13.

Fig. 11. Waves of order $n = 1$. Curve of the component $u_1^{(1)}$ for $r \gg a$; $\nu = 1$, $\xi = 1$.

Fig. 12. Waves of order $n = 1$. Curve of the component $u_2^{(2)}$ for $r \gg a$; $\nu = 1$, $\xi = 1$.

Fig. 13. Three-dimensional graph for waves of order $n = 1$. The curve of the radial component u_1; $\nu = 1$, $\xi = 1$.

und

$$S_1 = \omega_1^2 - \omega_1'^2, \qquad S_1' = 2\omega_1\omega_1',$$
$$S_2 = \omega_1 - \omega_2, \qquad S_2' = \omega_1',$$
$$S_3 = \omega_1 - \omega_3, \qquad S_3' = \omega_1'.$$

For $\nu = 1$ expressions (31) reduce to

$$u_{0j}^{(i)'} = \frac{\omega^2\beta_2 C_j^{(i)} e^{-\eta\tau_i'}}{\mu R^2\eta^3\Omega_{01}(-\eta)}\left\{\left(\eta\tau_i' + 2 + \frac{\eta Q_1}{\Omega_{01}(-\eta)}\right)\cdot (\mathfrak{a}_j^{(i)}R^2 + \mathfrak{b}_j^{(i)}R + \mathfrak{c}_j^{(i)}) - \eta(\mathfrak{A}_j^{(i)}R^2 + \mathfrak{B}_j^{(i)}R + \mathfrak{C}_j^{(i)})\right\}, \qquad T_i > 0, \tag{33}$$

$$u_{0j}^{(i)'} = 0, \qquad T_i < 0,$$

where

$$Q_1 = 2(\eta + \omega_1)(\eta + \omega_2)(\eta + \omega_3) + (2\eta + \omega_2 + \omega_3)\left[(\eta + \omega_1)^2 + \omega_1'^2\right],$$
$$\mathfrak{a}_1^{(1)} = -\eta^5 + 3\eta^4 - 6\eta^3 + 6\eta^2,$$
$$\mathfrak{b}_1^{(1)} = 2\sqrt{3}\eta^4 - 6\sqrt{3}\eta^3 + 12\sqrt{3}\eta^2 - 12\sqrt{3}\eta,$$
$$\mathfrak{c}_1^{(1)} = -6\eta^3 + 18\eta^2 - 36\eta + 36,$$
$$\mathfrak{A}_1^{(1)} = 4\eta^4 - 12\eta^3 + 18\eta^2 - 12\eta,$$
$$\mathfrak{B}_1^{(1)} = -8\sqrt{3}\eta^3 + 18\sqrt{3}\eta^2 - 24\sqrt{3}\eta + 12\sqrt{3},$$
$$\mathfrak{C}_1^{(1)} = 18\eta^2 - 36\eta + 36,$$

$$\mathfrak{a}_2^{(1)} = \mathfrak{A}_2^{(1)} = 0,$$
$$\mathfrak{b}_2^{(1)} = \eta^4 - 3\eta^3 + 6\eta^2 - 6\eta, \qquad \mathfrak{B}_2^{(1)} = -4\eta^3 + 9\eta^2 - 12\eta + 6,$$
$$\mathfrak{c}_2^{(1)} = -\sqrt{3}\eta^3 + 3\sqrt{3}\eta^2 - 6\sqrt{3}\eta + 6\sqrt{3}, \qquad \mathfrak{C}_2^{(1)} = 3\sqrt{3}\eta^2 - 6\sqrt{3}\,\eta + 6\sqrt{3},$$

$$\mathfrak{a}_1^{(2)} = \mathfrak{A}_1^{(2)} = 0,$$
$$\mathfrak{b}_1^{(2)} = -\eta^3 + 3\sqrt{3}\eta^2 - 9\eta, \qquad \mathfrak{B}_1^{(2)} = 3\eta^2 - 6\sqrt{3}\eta + 9,$$
$$\mathfrak{c}_1^{(2)} = \eta^2 - 3\sqrt{3}\eta + 9, \qquad \mathfrak{C}_1^{(2)} = -2\eta + 3\sqrt{3},$$

$$\mathfrak{a}_2^{(2)} = \eta^4 - 3\sqrt{3}\eta^3 + 9\eta^2, \qquad \mathfrak{A}_2^{(2)} = -4\eta^3 + 9\sqrt{3}\eta^2 - 18\eta,$$
$$\mathfrak{b}_2^{(2)} = -\eta^3 + 3\sqrt{3}\eta^2 - 9\eta, \qquad \mathfrak{B}_2^{(2)} = 3\eta^2 - 6\sqrt{3}\eta + 9,$$
$$\mathfrak{c}_2^{(2)} = \eta^2 - 3\sqrt{3}\eta + 9, \qquad \mathfrak{C}_2^{(2)} = -2\eta + 3\sqrt{3}.$$

For the forced waves of the first kind (30) and for the free wave the shape of the corresponding expressions for $\nu = 1$ is evident; we shall therefore not write them down explicitly.

The numerical results for both the components of the forced waves $u_{0j}^{(i)}$, $u_{0j}^{(i)'}$, the components of the free wave $u_{\nu j}^{(i)}$ and the components of the resultant displacements on the surface of the spherical source $r = a$ for $\nu = 1$, $\xi = 1$ ($\eta = \sqrt{3}$) are given in Figs. 5 to 8. From these figures it is evident that the components of the forced wave of the first kind grow with increasing time beyond bounds while the components of the forced wave of the second kind and the components of the free wave converge to zero. For the above reason the resulting displacement $u_j^{(i)}$ also diverges with increasing time. This circumstance is caused by the fact that the dilatational and shear waves were considered separately. However, since $u_{0j}^{(1)} + u_{0j}^{(2)} = 0$ the total radial component of the displacement u_1 and the tangential component of the displacement u_2 is finite. Their dependence as well as the dependence of $u_{0j}^{(1)'} + u_{0j}^{(2)'}$, $u_{0j}^{(1)} + u_{0j}^{(2)}$ on time is given in Figs. 9 and 10.

At great distance from the source $r \gg a$ the longitudinal radial component of the displacement $u_1^{(1)}$ behaves as $1/R$ while the transverse radial component $u_1^{(2)}$ behaves as $1/R^2$. For the tangential components of displacement on the other hand the longitudinal component u_2^{1} behaves as $1/R^2$ while the transverse component $u_2^{(2)}$ as $1/R$. Thus at sufficiently great distances from the source the transverse radial component $u_1^{(2)}$ and the longitudinal tangential component $u_2^{(1)}$ become negligible and the radial component has the character of a purely dilatational wave while the tangential component has the character of a transverse wave. The time dependence of these components $(u_1^{(1)}, u_2^{(2)})$ for $r \gg a$ is shown in Figs. 11 and 12.

The three-dimensional diagram for u_1 (Fig. 13) which represents both time and space dependence shows that conditions are more complicated for waves of order $n = 1$ than for waves of order $n = 0$. Of greatest interest is the fact that even for $R = 2$ the original simple maximum splits up into two quite distinct maxima. For $R = 3$ the second maximum already exceeds the first in magnitude. The splitting up is caused on the one hand by the transverse component $u_1^{(2)}$ and on the other hand by the longitudinal forced waves of the first kind $u_{01}^{(1)}$ the increase of which causes an increase of the displacement up to the moment of the arrival of the transverse waves, when $u_{01}^{(1)}$ and $u_{01}^{(2)}$ cancel.

It is evident from Figs. 9 to 13 that the waves of order $n = 1$ are not periodic. This is caused by roots of the expression $\Omega_{01}(z)$ as will be shown in the following paragraph.

3. The Physical Interpretation of the Roots of $\Omega_{0n}(z)$.

The free waves of order $n = 0$ [see (23)] represent a damped harmonic motion of the medium originating independent of the shape of the exciting function $f(t)$. The substance of the periodicity of the free waves lies in the fact that the roots of the expression $\Omega_{00}(z)$ are complex. As **Kawasumi** and **Yosiyama** [11] have pointed out it shows formal analogy with the forced oscillations of an inertial system. The solution of the differential equation of motion for an inertial system (x is displacement)

$$\ddot{x} + 2\varkappa\dot{x} + n^2 x = f(t) \tag{34}$$

can be written if the initial conditions are $x = 0$, $\dot{x} = 0$ for $t = 0$ in the form

$$x = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{g(z)\, e^{zt}}{z(z - z_1')(z - z_2')}\, dz\,, \tag{35}$$

where $z_{1,2}' = -\varkappa \pm i(n^2 - \varkappa^2)^{1/2}$ and $g(z)$ is the image of the function $f(t)$. The physical meaning of $\varkappa$ is the damping of the system, $2\pi/n$ the free period of the undamped system and $2\pi/(n^2 - \varkappa^2)^{1/2}$ the period of the damped system. For the case of elastic waves we get for the displacement u_1 [see Equ. (19)]

$$u_1 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{F(z)\, g(z)\, e^{zT_1}}{z(z - z_1)(z - z_2)}\, dz\,, \tag{36}$$

where $z_{1,2} = (a\beta_1)^{-1}(\omega \pm i\omega')$ are the roots of equation (21). Expression (36) is formally quite analogous to expression (35). In the numerator of the integrand the variable T_1 corresponds to the time t and both expressions differ only by the function $F(z)$, which, however, neither affects the period nor the damping of the free waves. The denominator of both expressions is formally identical. This means that the real part of $z_{1,2}$ corresponds to $\varkappa$, the imaginary part to $(n^2 - \varkappa^2)^{1/2}$. We can therefore interpret $-\omega/a\beta_1$ as the damping and $2\pi a\beta_1/\omega'$ as the period corresponding to the system of the spherical cavity of radius a in an infinite elastic medium characterized by the parameter β_1.

Generalizing this above interpretation for waves of arbitrary order, we are entitled to denote the expressions due to those poles of the integrand of the Bromwich-Wagner integral which are the roots of the equation

$$\Omega_{0n}(z) = 0\,, \tag{37}$$

as the *free waves* and the expressions due to the other poles depending without exception on the character of the exciting function as *forced waves*. The real parts of the roots of (37) represent the damping, the imaginary parts $\frac{1}{2\pi}$ times the frequency of the system of the spherical cavity in an infinite elastic medium. Both these quantities, the same as the damping and the period of the characteristic waves depend on the radius of the cavity a and on the parameters of the medium β_i.

Now it is quite clear why waves of order $n = 1$ are aperiodic. This is caused by the fact that the effect of the real roots of Equ. (27) completely covers the effect of the remaining pair of complex conjugated roots. The result is that for waves of order $n = 1$ the system of the spherical cavity in an infinite elastic medium behaves like a system aperiodically damped.[8])

IV. ANALYSIS OF RESULTS

1. Decrease of Amplitude with the Distance.

In this paper we shall understand under the amplitude A the magnitude of the first maximum of displacement. Let us now determine the decrease of the amplitude with increasing distance from the source. This dependence cannot be expressed directly in an analytic form. Diagrams of displacement versus time for a sufficiently dense series of distances were therefore plotted from which it is possible to read off the value of the maximum displacement with sufficient precision. The graphical dependence between the amplitude and the distance was thus obtained. The method is very tedious.

The dependence was investigated for waves of order $n = 0$ for various values of the parameter α (or ξ) of the exciting function. The exciting function was taken to be of the form

$$f(t) = \sigma_0 \alpha^\nu t^\nu e^{-\alpha t}.$$

Its maximum $f_{max} = \sigma_0(\nu/e)^\nu$ is independent of α meaning that for the same ν the maximum stress at surface of the spherical source is identical for all α (see Fig. 1a) and that the displacements are comparable.

[8]) Waves of order $n = 2$ are again periodic (this holds quite generally for all even orders). The aperiodicity of odd order waves is probably not general since the preliminary investigation of waves of order $n = 3$ rather points to their periodic character.

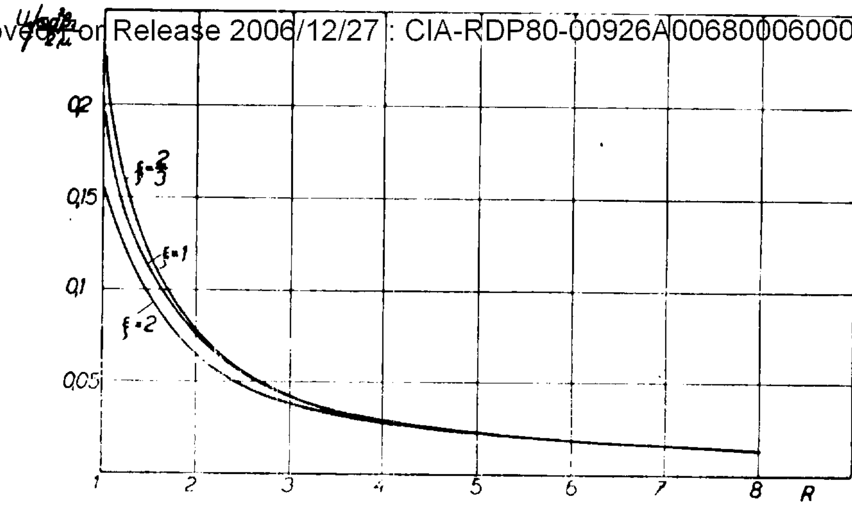

Fig. 14. Dependence of the amplitude on the distance for waves of order $n = 0$.

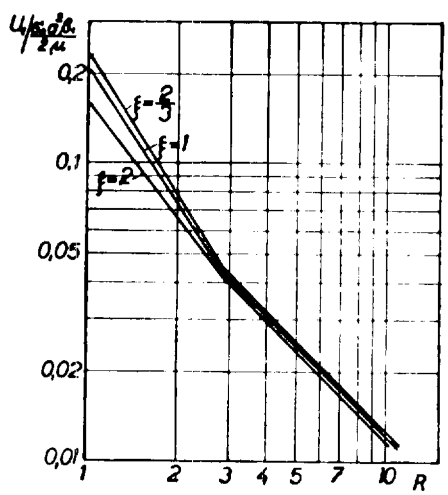

Fig. 15. Dependence of the amplitude on the distance for waves of order $n = 0$.

For a given $a\beta_1$ the quantity ξ changes only with ν. The results for $\nu = 1$ and $\xi = \frac{2}{3}$, 1, 2 are given in Fig. 14. By a transformation of the relation between A and R into the system ($\log A$, $\log R$) the curves for each ξ from Fig. 14 fall into two parts which can be approximated by two straight lines having different slope (see Fig. 15). The amplitude is therefore

$$A = \frac{A_0}{R^k}. \tag{38}$$

For $R > 3$, $k = 1$. For $R < 3$, k is a function of the parameter ξ; to the values of $\xi = \frac{2}{3}$, 1, 2 there correspond $k = 1.56$, 1.52, 1.27 respectively. A_0 is also

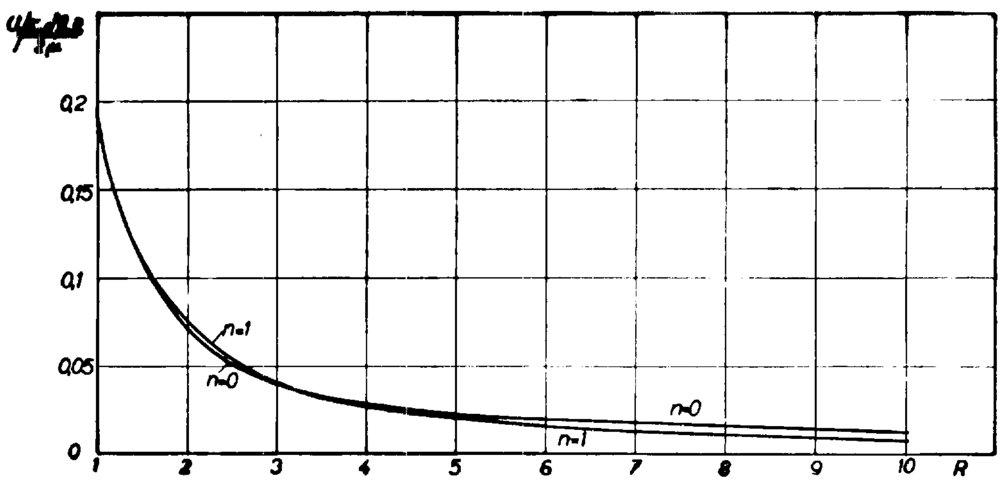

Fig. 16. Dependence of the amplitude on the distance for waves of order $n = 1$.

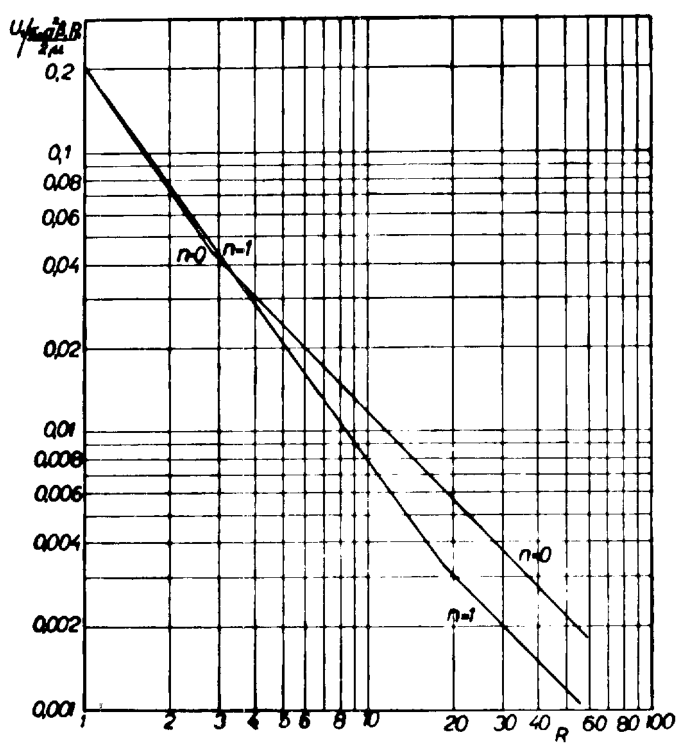

Fig. 17. Dependence of the amplitude on the distance for waves of order $n = 1$.

a function of ξ. From this it is evident that the effect of the exciting function on the manner of propagation of waves of order $n = 0$ is practically limited to $r < 3a$.

In a similar manner waves of order $n = 1$ were investigated where, however, only the radial component u_1 has been studied up to the present. The dependence of the amplitude A on the reduced distance R for this case is shown in Fig. 16, for the parameters $\nu = 1$, $\xi = 1$. For comparison the curves for $n = 0$ are also given in the same figure. In the system ($\log A$, $\log R$) the curve again falls into two linear parts having different slope, so that (38) again holds. Now $k = 1.41$ for $R < 19$ and $k = 1$ for $R > 19$ (see Fig. 17). The effect of the exciting function reaches here more than six times the distance of the case $n = 0$. For the distance $r > 19a$ the amplitudes of waves of order $n = 1$ are 1.9 times smaller than for waves of order $n = 0$.

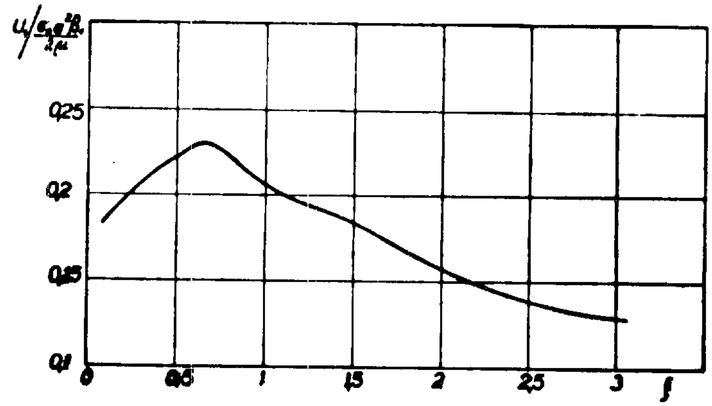

Fig. 18. Dependence of the amplitude on the parameter ξ for waves of order $n = 0$.

2. The Dependence on the Parameters of the Exciting Function.

The dependence of the amplitude on the parameters of the exciting function ν and α was investigated for waves of order $n = 0$. The dependence on ν for $R = 1$ and $\alpha = 1$ is evident from the following table. To normalize the stress on the surface of the source independent of ν the exciting function was given the form

$$f(t) = \sigma_0 c_\nu t^\nu e^{-\alpha t},$$

where c_ν was determined so that $c_1 = 1$ (see Fig. 1b)

ν	c_ν	$u_1/\frac{\sigma_0}{\mu}$
0	0.368	$0.063a$
1	1	$0.102a^2\beta_1$
2	0.680	$0.115a^3\beta_1^2$
3	0.274	$0.108a^4\beta_1^3$

The deciding factor here is the quantity β_1 which according to (7′) is the inverse of the velocity of the longitudinal waves. Its magnitude is of the order of 10^{-5} to 10^{-6} in absolute units. This means that for increasing ν the amplitude very rapidly decreases.

The dependence on the parameter α (or ξ) is shown in Fig. 18 for $\nu = 1$ and $R = 1$. The curve is similar in character to the known resonance curve. The maximum occurs for $\xi = \frac{2}{3}$ which is the value of the damping for a spherical cavity in an infinite elastic medium. Here again the phenomenon is formally analogous to resonance in a damped inertial system [12]. Resonance effects for waves of higher orders are much more complicated[9]) and are not studied in this paper.

V. CONCLUSION

The above paper is devoted to the investigation of elastic waves produced in an infinite homogeneous isotropic and perfectly elastic medium by a spherical source if the stress on its surface is in general an arbitrary function of time. A general solution of the problem was found. Next, waves of order $n = 0$ and $n = 1$ produced by a shock function of the type $\sigma t^{\nu} e^{-\alpha t}$ were investigated. On analysis of the results it was shown that in a certain vicinity of the source ($r < 3a$ for $n = 0$ and $r < 19a$ for $n = 1$) the amplitude does not decrease according to the law $1/r$ but according to $1/r^k$ where $k > 1$. In the neighbourhood of the source the amplitude also depends on the parameters of the exciting function. The significance of the results in applied seismology will be the subject of another paper.

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К ТЕОРИИ УПРУГИХ ВОЛН ВЫЗВАННЫХ УДАРОМ

(Содержание предыдущей стати)

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Исследуются упругие волны вызванные шаровым источником в бесконечно распространенной однородной, изотропной и совершенно упругой среде, при чем предполагается, что напряжение распределенное сфероидально на поверхности источника является произвольной функцией времени (возбуждающая функция). Ввиду сложности вычислений пренебрегается азимутальной составляющей смещения и рассматривается только двухразмерная проблема в плоскости $\vartheta =$ конст., решение которой основано на решении уравнений (1a, b) с учетом условий (3), (4) и (D), и которые с помощью преобразования *Лапласа* сводятся к вычислению интегралов *Бромвич-Вагнера* (17a, b, c, d). Ввиду некоторой его важности для практической сейсмологии, был вычислен и подвергся обсуждению случай источника взрывного характера, который характеризуется граничными условиями (3′), при чем предполагается, что возбуждающая функция $f(t)$

[9]) The phenomenon is again connected with the zeros of expression (37).

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определяется выражением (18a) и ход изменения каковой показан на рис. 1a, b. Подробно исследуются волны порядка $n = 0$ и $n = 1$. Решение для $n = 0$ дано выражением (23), в котором первый член в составных скобках представляет вынужденную составляющую волны а второй член — собственную волну (причина которой — полюсные значения подинтегрального выражения (20), удовлетворяющие уравнению (21). Для частного случая $\nu = 1$ (см. 18a) имеет место уравнение (24), для значительного же расстояния от источника справедливо уравнение (25). Ход изменения смещения указан на рис. 2, 3 и на трехразмерной диаграме рис. 4. Несмотря на то, что возбуждающая функция апериодическая, результирующая волна будет затухающей периодической волной. У волн порядка $n = 1$ решение для вынужденных волн первого рода (неэкспоненциальная составляющая вынужденных волн) дано выражением (30), для вынужденных же волн второго рода (экспоненциальная составляющая вынужденных волн) справедливо выражение (31), а для собственных волн выражение (32). Для частного случая $\nu = 1$ вынужденные волны второго рода определяются выражением (33). Результаты вычислений представленны на рисунках 5—13. Волны порядка $n = 1$ будут апериодическими и расходящимися (в силу свойства вынужденных волн первого рода расходиться), если рассматриваются отдельно продольные и поперечные части (смотри рис. 5—8). Но результирующая радиальная и тангенциальная составляющая смещения имеет конечную величину (см. рис. 9 и 10). Далее обращается внимание на аналогию между исследуемой проблемой и вынужденными колебаниями инертной системы. Обобщается понятие собственных и выниужденных волн и для более высокого их порядка и объясняется сущность периодичности или апериодичности волн, которая заключается в характере нулевых точек выражения (37). В подробном обсуждении исследуется зависимость, согласно которой уменьшается амплитуда (первый максимум смещения) волн при возрастании расстояния от источника. Зависимость выражается соотношением (38), где для волн порядка $n = 0$ будет $k = 1$ при $R > 3$ и $k > 1$ при $R < 3$ (см. рис. 14 и 15; здесь R — приведенное расстояние от источника возбуждения, см. (22)); для волн же порядка $n = 1$ будет $k = 1$ при $R > 19$ и $k > 1$ при $R < 19$ (смотри рис. 16 и 17). Амплитуда также зависит от параметров возбуждающей функции (см. таблицу стр. 116; на рис. 18 видны начинающиеся резонансные явления).

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